

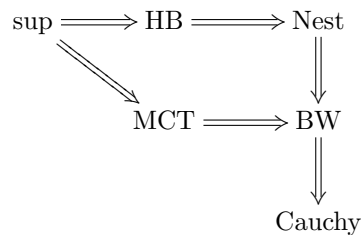
Assignment 3.

This homework is due *Thursday*, September 24.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 11.

1. EXERCISES

- (1) (1.4.36) Show that the Borel σ -algebra \mathcal{B} is the smallest σ -algebra \mathcal{A} that contains all intervals of the form $[a, b)$, where $a < b$. (*Hint*: Show that *both* \mathcal{B} and \mathcal{A} contain *both* open intervals and intervals of the form $[a, b)$.)
- (2) Show that the Heine–Borel theorem is false for:
 - (a) open covers of an open bounded set. (That is, give an example of an open bounded set and its open cover for which the conclusion of the Heine–Borel theorem fails.)
 - (b) open covers of a closed unbounded set.
- (3) In lectures, the following implications were proved or at least sketched (sup = Completeness Axiom, HB = Heine–Borel Theorem, Nest = Nested Set Theorem, BW = Bolzano–Weierstrass Theorem, MCT = Monotone Convergence Theorem):



Prove enough implications to make the top five statements equivalent to each other. (Don't forget that you have to prove Archimedean Principle if you use it.)

- (4) An extended real number c (that is, $c \in \mathbb{R}$, or $c = \infty$, or $c = -\infty$) is called a *cluster point* (or a limit point) of a sequence $\{a_n\}$ if a subsequence of $\{a_n\}$ converges¹ to c . Show that the set of all real cluster points of a sequence in \mathbb{R} is a closed set.
- (5) (\sim 1.5.40) Prove that a sequence in \mathbb{R} converges to an extended real number $a \in \mathbb{R}$ if and only if the set of (extended real) cluster points of this sequence is the singleton $\{a\}$.

2. EXTRA EXERCISE

- (6) Prove or disprove. For any closed set $F \subseteq \mathbb{R}$, there is a sequence in \mathbb{R} whose set of real cluster points is precisely F .

¹A sequence $\{a_n\}$ converges to ∞ if $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N, a_n > \varepsilon$. Similarly for $-\infty$.